Entanglement and quantum phase transitions in matrix product spin one chains

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Abstract

We consider a one-parameter family of matrix product states of spin one particles on a periodic chain and study in detail the entanglement properties of such a state. In particular we calculate exactly the entanglement of one site with the rest of the chain, and the entanglement of two distant sites with each other and show that the derivative of both these properties diverge when the parameter g of the states passes through a critical point. Such a point can be called a point of quantum phase transition, since at this point, the character of the matrix product state which is the ground state of a Hamiltonian, changes discontinuously. We also study the finite size effects and show how the entanglement depends on the size of the chain. This later part is relevant to the field of quantum computation where the problem of initial state preparation in finite arrays of qubits or qutrits is important. It is also shown that entanglement of two sites have scailing behavior near the critical point.

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1 Introduction

Interacting spin systems are among the most promising candidates for the actual implementation of quantum computers in the future. In the new terminology which has emerged since the upsurge of interest in quantum computation, a spin 1/2 system refers not only to the actual spin degrees of freedom of an atom or nucleus, but it refers to any system, with any number of levels, in which we have selected two states for encoding the information. For example the ground state and the first excited state of an ion in an ion trap, make a spin 1/2 system or a so called qubit. Similar terminology is used for three level states or qutrits. These latter systems and their generalizations to d- level states or qudits are of immense interest, since it is not yet clear if actual implementation of quantum computers will be based on two level systems. In view of this, many systems which have been traditionally the focus of study in condensed matter physics are being examined from different points of view related to quantum computation and information. The most important property of an interacting spin system, which is of relevance to this new emerging field, is entanglement or non-local quantum correlation. In fact it is considered as a resource, like energy, since it plays a vital role in any process of quantum information and computation, moreover it can be measured, manipulated and transferred. Consider the ground state of an interacting spin system, comprising N spins. There are some basic questions regarding entanglement: How much two distant spins are entangled with each other? How this entanglement varies with the system size N? Is there any threshold distance, beyond which there will be no entanglement at all? How the entanglement varies when we approach a point of quantum phase transition? Answering these questions requires tools which have been developed only recently in the field of quantum information [1, 2, 3, 4, 5, 6]. In this way a fruitful field of investigation at the borderline of condensed matter physics and quantum information has emerged. The aim of this letter is to investigate such questions for a class of spin-one states, which are known to be exact ground states of certain multi-parameter families of Hamiltonians describing nearest neighbor interactions of spin-one particles on a periodic chain. The method for construction of such states, known also as the matrix product formalism, was first introduced in [7, 8, 9] and then applied to various models in [10, 11, 12, 13]. Recently it has been applied even to two dimensional models [14, 15]. Such states can be constructed to have specified symmetry properties or even to induce certain kinds of quantum phase transitions with pre-determined properties [16]. The entanglement properties of the so called AKLT models [7], which inspires the matrix product states was first studied in [17]. Here we study the entanglement properties of a one parameter deformation of AKLT models.

We first study the entanglement of one site with the rest of the chain and the entanglement of two distant sites in the thermodynamic limits, however we mainly focus on states with finite but arbitrary number of spins, since this is the case of interest from the point of view of quantum computation and information. We will determine the entanglement of two spins in the lattice as a function of their distance, the coupling constant of the state or the Hamiltonian, which we denote by g, and the system size. The results are that: 1- When the parameter g approaches its critical value $g_0 = 0$,

the range of entanglement increases indefinitely at the cost of its magnitude, fig. (1), 2- for any non-zero value of g, there is a threshold distance beyond which there is no entanglement between spins, fig. (3), 3- for any two spins with a fixed distance, there is a threshold system size, above which entanglement vanishes, fig. (4), and there is a scaling behaviour in the entanglement of two adjutant spins which is shown in figure (7).

Let us first remind the matrix product formalism in a language which we find convenient.

2 Matrix Product States

For a homogeneous ring of N sites, where each site describes a d-level state. The Hilbert space of each site is spanned by the basis vectors $|i\rangle$, $i = 0, \dots d - 1$. A state

$$|\Psi\rangle = \sum_{i_1, i_2, \dots i_N} \psi_{i_1 i_2 \dots i_N} |i_1, i_2, \dots, i_N\rangle \tag{1}$$

is called a matrix product state, if there exist matrices A_i , $i = 0 \cdots d-1$ (of dimension D) such that

$$\psi_{i_1 i_2 \cdots i_N} = \frac{1}{\sqrt{Z}} tr(A_{i_1} A_{i_2} \cdots A_{i_N}),$$
 (2)

where Z is a normalization constant equal to $Z = tr(E^N)$ and $E := \sum_{i=0}^{d-1} A_i^* \otimes A_i$. The correlation functions are readily calculated in this formalism. For example, for the one-point functions we have

$$\langle \Psi | O_k | \Psi \rangle = \frac{tr(E^{k-1} E_O E^{N-k})}{tr(E^N)},\tag{3}$$

where $E_O = \sum_{i,j} \langle i|O|j\rangle A_i^* \otimes A_j$. In the thermodynamic limit, only the largest eigenvalue of E survives and so any level crossing in the largest eigenvalue of E, leads to a discontinuity of correlation functions. This may be termed an MPS-quantum phase transition [16].

2.1 Gauge Transformations

From (2), it is evident that two sets of matrices $\{A_i\}$ and $\{A'_i\}$ lead to the same matrix product state if they are related as $A'_i = \mu S A_i S^{-1}$, where μ is a scale factor and S is any invertible matrix. Actually the gauge transformation can be more general than this, namely $A'_i = \mu S A_i S'$ with S'S = I. Such transformations can be used to gauge away irrelevant parameters in the matrices A_i .

2.2 Symmetries

On a ring, the state (1) is invariant under translation. Demanding more symmetries imposes constrains on the matrices A_i . Considering equation (2), the state is symmetric under parity if there exists a matrix Π such that

$$A_i^T = \sigma \Pi A_i \Pi^{-1} \qquad \sigma = \pm 1 \tag{4}$$

where A^T is the transpose of A and it has time reversal symmetry if the matrices A_i are real.

Consider now a local symmetry operator R acting on a site as $R|i\rangle = R_{ji}|j\rangle$ where summation convention is being used. R is a d dimensional unitary representation of the symmetry. A global symmetry operator $\mathcal{R} := R^{\otimes N}$ will then change this state to another matrix product state

$$\Psi_{i_1 i_2 \cdots i_N} \longrightarrow \Psi' := tr(A'_{i_1} A'_{i_2} \cdots A'_{i_N}), \tag{5}$$

where $A'_i := R_{ij}A_j$. The state $|\Psi\rangle$ is invariant under this symmetry if there exists an operator U(R) such that

$$R_{ij}A_j = U(R)A_iU^{-1}(R).$$
 (6)

Thus R and U(R) are two unitary representations of the symmetry, respectively of dimensions d and D. In case that R is a continuous symmetry with generators T_a , equation (6), leads to

$$(T_a)_{ij}A_j = [T_a, A_i], (7)$$

where T_a and T_a are the d- and D-dimensional representations of the Lie algebra of the symmetry.

2.3 The Hamiltonian

To construct a Hamiltonian with nearest neighbor interaction, in a way that the state in equation (1) be its ground state, we have to find the null space of the density matrix of two adjacent sites which is given by:

$$\rho_{ij;kl} = \frac{tr((A_i^* A_j^* \otimes A_k A_l) E^{N-2})}{tr(E^N)}.$$
(8)

The null space of this reduced density matrix include the solutions of

$$\sum_{k,l}^{d-1} c_{kl} A_k A_l = 0. (9)$$

The number of independent solutions of this system of equation is $d^2 - D^2$. Thus for this density matrix to have a null space it is sufficient that d > D. Let the null space of the reduced density matrix be spanned by the orthogonal vectors $|e_{\alpha}\rangle$, $(\alpha = 1, \dots s \ge d^2 - D^2)$. Then we can construct the local hamiltonian acting on 2 consecutive sites as $h := \sum_{\alpha=1}^{s} \lambda_{\alpha} |e_{\alpha}\rangle \langle e_{\alpha}|$, where λ_{α} 's are non-negative constants. These parameters together with the parameters of the vectors $|e_{\alpha}\rangle$ inhertited from those of the original matrices A_i , determine the total number of coupling constants of the Hamiltonian. The full Hamiltonian on the chain is written as $H = \sum_{l=1}^{N} h_{l,l+1}$, where $h_{i,i+1}$ is the embedding of h into sites i and i+1. The state $|\Psi\rangle$ is then a ground state of this Hamiltonian with vanishing energy (for an exposition see [15]).

3 Matrix Product States on Spin 1 Chains

3.1 Construction of the state

The matrix product state, that we use for spin-one systems, has already been constructed in [10]. Here we review its construction in the language introduced in previous section for completeness. Since d=3, to guarantee a null space for the two-site density matrix, we set D=2. The matrices A_1 , A_0 and $A_{\overline{1}} \equiv A_{-1}$ correspond to the local states $|1\rangle$, $|0\rangle$ and $|\overline{1}\rangle \equiv |-1\rangle$ respectively, where $S_z|m\rangle = m|m\rangle$. Considering equation (7), the symmetry around the z axis requires that

$$[S_z, A_m] = mA_m, (10)$$

where $S_z = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Solving (10), demanding parity symmetry (4) and getting rid of irrelevant parameters with suitable gauge transformations, leave us with:

$$\mathcal{A} = \begin{pmatrix} |0\rangle & -\sqrt{2}g|1\rangle \\ |\overline{1}\rangle & \sigma|0\rangle \end{pmatrix},\tag{11}$$

where we have used the compact notation $\mathcal{A}:=\sum_i A_i|i\rangle$. Note that the state constructed in this way automatically has spin-flip symmetry, i.e. $XA_mX^{-1}=\sigma A_{\overline{m}}$ with $X=\begin{pmatrix} 0 & -\sigma g \\ 1 & 0 \end{pmatrix}$. Also at $(g,\sigma)=(1,-1)$, the so-called AKLT point [7], the model has full rotational symmetry, since in this case the states $-\sqrt{2}\sigma_+,\sigma_z$ and $\sqrt{2}\sigma_-$, form a vector under the adjoint representations of the rotation group.

3.2 The Hamiltonian

As mentioned previously, we must solve (9) for matrices (11) to construct the hamiltonian. It is straightforward to verify that the solution space of (9) is spanned by the following vectors:

$$|e_{1}\rangle = |1,1\rangle$$

$$|e_{2}\rangle = \frac{1}{\sqrt{2}}(|1,0\rangle - \sigma|0,1\rangle)$$

$$|e_{3}\rangle = \frac{1}{\sqrt{2+4g^{2}}}(|1,\overline{1}\rangle + 2g|0,0\rangle + |\overline{1},1\rangle)$$

$$|e_{4}\rangle = \frac{1}{\sqrt{2}}(|0,\overline{1}\rangle - \sigma|\overline{1},0\rangle)$$

$$|e_{5}\rangle = |\overline{1},\overline{1}\rangle.$$
(12)

With these vectors, we write the local Hamiltonian as

$$h = a(|e_1\rangle\langle e_1| + |e_5\rangle\langle e_5|) + b(|e_2\rangle\langle e_2| + |e_4\rangle\langle e_4|) + c|e_3\rangle\langle e_3|, \tag{13}$$

to preserve the Z_2 symmetries mentioned above. Writing this in terms of local spin operators, the final form of H is obtained as:

$$H = \sum_{i=1}^{N} J_1 \mathbf{S}_i \cdot \mathbf{S}_{i+1} + J_2 (\mathbf{S}_i \cdot \mathbf{S}_{i+1})^2 + J_3 S_{z,i} S_{z,i+1} + J_4 (S_{z,i} S_{z,i+1})^2 + J_5 S_{z,i}^2 + J_6 \{ \mathbf{S}_i \cdot \mathbf{S}_{i+1}, S_{z,i} S_{z,i+1} \}_+,$$
(14)

where

$$J_{1} = -b\sigma(1+2g^{2}), J_{2} = c,$$

$$J_{3} = (a+b\sigma)(1+2g^{2}), (15)$$

$$J_{4} = (a+2b(\sigma-1))(1+2g^{2}) + (1+2g)^{2}c,$$

$$J_{5} = 2b(1+2g^{2}) + 2c(1-4g^{2}),$$

$$J_{6} = -b\sigma(1+2g^{2}) - c(1+2g).$$

In writing the above Hamiltonian we have ignored an overall additive constant and have re-scaled the operator (13) by a constant $2(1+2g^2)$. This represent a four parameter family of Hamiltonians which have (1) as their ground state. Full rotational symmetry exists ,when we have $(g,\sigma)=(1,-1)$, and a=b=c=1, for which case the Hamiltonian is known as the AKLT model.

4 The thermodynamic limit

We can derive many properties of the ground state using the transfer matrix formalism explained in section (2). The eigenvalues of the matrix E are found to be

$$\lambda_1 = 1 + 2g$$
 $\lambda_2 = 1 - 2g$ $\lambda_{3,4} = \sigma.$ (16)

This shows a level crossing in the largest eigenvalue of E and hence a singularity in correlation functions at g=0. The average magnetization and the correlation functions are found to be [10]

$$< S_x^i > = < S_y^i > = < S_z^i > = 0,$$

and

$$\begin{split} & < S_z^1 S_z^r > = -\frac{4g^2}{(1-2|g|)^2} \left(\frac{1-2|g|}{1+2|g|}\right)^r \\ & < S_{\mathbf{n}}^1 S_{\mathbf{n}}^r > = -2|g|(\sigma - Sign(g)) \left(\frac{\sigma}{1+2|g|}\right)^r, \end{split}$$

where \mathbf{n} is any unit vector in the xy plane. The longitudinal and transverse correlation length, diverges at g=0, It is a natural question to ask if the same thing happens for entanglement when g approaches this critical point. In [18], the one-site entropy which measures the entanglement of one site with the rest of the lattice and also the two-site entropy which measures the entanglement of these two sites with the rest of the lattice were calculated in the thermodynamic limit. Here we use the negativity to measure how much two distant spins are entangled with each other. Moreover, we study in detail finite size effects to see how various properties of entanglement depend on the system size.

4.1 Entanglement of two distant sites

In the ground state of (14), any two particles will be in a mixed state. The reduced density matrix of two spins located at sites 1 and r is denoted by $\rho^{1,r}$. The rotational symmetry around the z axis shows that $\rho_{ij;kl} = 0$ unless i + j = k + l. Also the parity symmetry entails the condition $\rho_{ij,kl} = \rho_{\overline{ij};\overline{kl}}$, where $\overline{i} = -i$. Straightforward calculation shows that

$$\rho(1,r) = \begin{pmatrix}
\alpha & & & & & & & & \\
& |g|\gamma & & \mu & & & & \\
& & \beta & & \delta & & \nu & & \\
& & & \beta & & \delta & & \nu & & \\
& & & & |g|\gamma & & & & \\
& & & & & |g|\gamma & & \mu & & \\
& & & \nu & & \delta & & \beta & & \\
& & & \mu & & |g|\gamma & & & \\
& & & & \alpha
\end{pmatrix}$$
(17)

in which

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{g^2(\lambda_1^{r-2} \mp \lambda_2^{r-2})}{\lambda_1^r}, \quad \nu = 0, \quad \gamma = \frac{1}{\Lambda_1^2},$$

$$\delta = -g\left(\frac{\sigma}{\Lambda_1}\right)^r, \quad \mu = \sigma|g|\left(\frac{\sigma}{\Lambda_1}\right)^r, \quad (18)$$

where $\Lambda_1 = 1 + 2|g|$ and $\Lambda_2 = 1 - 2|g|$. Since the state of two sites, is a mixed state, we can not use von Neumann entropy to measure the entanglement between these two sites. In [19] it is shown that the necessary condition for a mixed state ρ , to be separable is that its partial transpose has non-negative eigenvalues. The quantitative version of this criterion is *Negativity* which is defined as follows [20]:

$$\mathcal{E}(\rho(1,r)) = \frac{||\rho^{T_A}(1,r)||_1 - 1}{2} \tag{19}$$

where $\rho^{T_A}(1,r)$ is the partial transpose of $\rho(1,r)$ with respect to the subsystem A, and $||X||_1 = \sqrt{X^{\dagger}X}$ is the trace norm of X. Equivalently it is equal to the sum of absolute values of negative eigenvalues of the matrix $\rho^{T_A}(1,r)$. In other words by means of Negativity we can measure the degree to which the partial transpose of the state ρ fails to be positive or how far the state of the two particles is from a separable state. A basic property of Negativity is that it is an entanglement monotone, meaning that the more entangled a state, the more negative it is in the above sense.

The eigenvalues of $\rho^{T_A}(1,r)$ are found from (17) to be:

$$\begin{array}{rcl} \omega_1 & = & \alpha & \omega_{2,3} = \beta & \omega_{4,5} = |g|\gamma + \delta \\ \\ \omega_{6,7} & = & |g|\gamma - \delta, \quad \omega_{8,9} = \frac{\alpha + \gamma + \nu}{2} \pm \frac{1}{2} \sqrt{(\alpha - \gamma + \nu)^2 + 8\mu^2}. \end{array}$$

From these eigenvalues the negativity and hence the entanglement of two spins can be calculated. Figure (1) shows the entanglement of two spins, one at site 1 and the

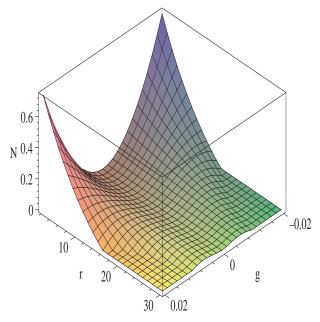


Figure 1: (Color Online) The entanglement of two spins at sites 1 and r as a function of their distance and the coupling g. Near the critical point the range of entanglement increases at the cost of its amplitude, for every g there is a maximum distant, beyond which there is no entanglement.

other at site r as a function of r and the parameter g. It is seen that in these models the range of entanglement increases as we approach the critical point, however its value decreases when we approach this point. We can obtain an approximate expression for the maximum range of entanglement. Inspection shows that the only eigenvalue of $\rho^{T_A}(1,r)$ which may go negative is

$$\omega_9 = \frac{\alpha + \gamma}{2} - \frac{1}{2}\sqrt{(\alpha - \gamma)^2 + 8\mu^2},$$

note that $\nu=0$ in the thermodynamic limit (18). Thus the inequality $\omega_9<0$ determines the range of entanglement. This is however equivalent to $\alpha\gamma<2\mu^2$ or in view of (18)

$$(1+2|g|)^{r-2} - (1-2|g|)^{r-2} < 2(1+2|g|)^{2-r}.$$

For small values of g, i.e. $|g| \ll 1$, we can write $1 \pm 2|g| \approx e^{\pm 2|g|}$ and then the above inequality transforms to

$$r \le \frac{\ln(3)}{4|g|} + 2,$$

implying that a entanglement exists up to a range of $r_0 \approx \frac{ln(3)}{4g} + 2$. As an example for g = 0.02, this gives $r_0 \approx 15$, which is also evident from figure (1).

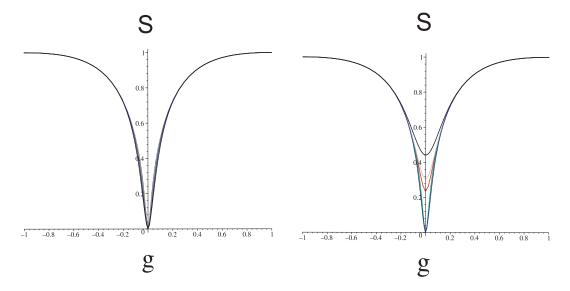


Figure 2: The single site entanglement for models with $\sigma = 1$ (left) and $\sigma = -1$ (right), for several values of system sizes, N=15, 21, 25, 31, and 35. For $\sigma = 1$, the entanglement is almost independent of size, as long as N > 10 and for $\sigma = -1$, it is so unless g is very close to the critical point.

5 Systems with finite size

The interest in entanglement properties of spin systems stems not only from its possible relation to the critical properties of such systems, but also from their possible candidacy for the future implementation of quantum computers. In this case we are dealing with a finite array of interacting spins which has relaxed to its ground state. Tuning the interactions of these spins with each other, changes the ground state and it is desirable to have controllable entanglement between different spins of this array. Therefore in this section we study more closely the properties of such matrix product states for finite values of N.

5.1 Single site entanglement

To measure the entanglement of one site with the rest of the chain we calculate the von Neumann entropy of the density matrix of one site, which is readily found to be

$$\rho^{(1)} = a(|1\rangle\langle 1| + |\overline{1}\rangle\langle \overline{1}|) + b|0\rangle\langle 0| \tag{20}$$

with

$$a = \frac{|g|[(1+2|g|)^{N-1} - (1-2|g|)^{N-1}]}{Z}, \qquad b = 1-2a$$
 (21)

and

$$Z = (1+2|g|)^{N} + (1-2|g|)^{N} + 2\sigma^{N}.$$
 (22)

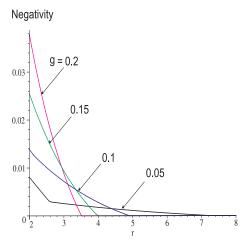


Figure 3: (Color Online) The entanglement of two distant spins as a function of their distance for a ring of size N=40 for several values of g near the critical point.

The one-site entropy which measures the entanglement of one site with the rest of the lattice will then be given by

$$S = -2a\log(a) - (1 - 2a)\log(1 - 2a). \tag{23}$$

For N >> 1 one can verify that $a \simeq g$, therefore for system sizes, S(g, N) is independent of N. Figure (2) shows the behavior of one-site entanglement for different system sizes. It shows clearly the difference between the models with $\sigma = 1$ and $\sigma = -1$. Note that the difference of $\sigma = -1$ and $\sigma = 1$ models shows up only for systems of odd size.

5.2 Entanglement of two distant sites

The entanglement of two sites can be measured by the negativity of the reduced density matrix of two sites. The general form of $\rho(1,r)$ is the same as in (17) except that the correlation functions now depend on the system size N in a rather complicated way. We do not write the explicit form of $\rho(1,r)$ for finite N and only report the behavior of entanglement in figures (3) and (4). Figure (3) shows the entanglement of two distant spins as a function of their distance for a ring of size N=40 for several values of g near the critical point. Several features are evident. First the entanglement has always a finite range. Furthermore, the range of entanglement increases as we approach the critical point, however its value decreases. Near the critical point, it is well known that correlation lengths diverge, here we see that entanglement range also diverges although this is accompanied by lowering of its value. Figure (4) shows entanglement between adjacent spins as a function of the size of the system at a fixed value of q. The figure shows that two spins at distance 4 (r=5) can be entangled for rings of size up to N=26. Also for any value of q and any distance r, there is a maximum system size $N_{max}(r,g)$ above which those two sites can not be entangled at all. This figure shows that $N_{max}(r,g)$ decreases with r.

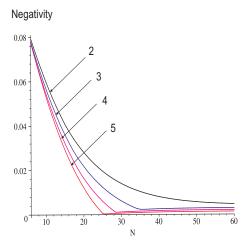


Figure 4: (Color Online) The entanglement of two spins with distances 1, 2, 3 and 4 (from top to bottom) as a function of the system size.

In figure (5) the entanglement of nearest neighbor sites is displayed for several values of system sizes when $\sigma=1$. At the critical point, the nearest neighbor spins are not entangled, however, entanglement appears for any infinitesimal deviation from this point. On the other hand when $\sigma=-1$ (the class of models which contain the AKLT point), there is entanglement at g=0, only for rings of odd size (figure 6). The behavior for even-sized rings is identical with the case when $\sigma=1$.

Figure (5) suggests a scaling behavior for entanglement near the critical point. To investigate this property, we consider for definiteness, the case $\sigma = 1$. For each value of N, the entanglement attains a maximum at a point $g_m(N)$, where its value at this point is denoted by $\mathcal{E}_m(N)$. In the inset of figure (7) we plot $log(g_m)$ and $log(\mathcal{E}_m)$ versus log (N). We find numerically that

$$\log(g_m) = -1.077 \log(N) - 0.106 \approx -\log N - c, \log(\mathcal{E}_m) = -1.086 \log N - 0.214 \approx -\log N - d,$$
(24)

where c and d are two constants, independent of N. From the scaling hypothesis, this means that one can write the negativity in the vicinity of the critical point as

$$\mathcal{E}(g,N) \approx \frac{1}{N} f(Ng),$$

where f is a universal function. In figure (7) we plot $\log(N\mathcal{E})$ as a function of $\log(Ng)$. It shows that all the data collapse on a single curve for 4 order of magnitudes of gN.

5.3 Limiting form of the states

It is instructive to find the explicit form of the state near the critical point, i.e. when $|g| \ll 1$. For such an analysis we should find the dominant amplitudes $\psi_{i_1 \cdots i_N}$ in the

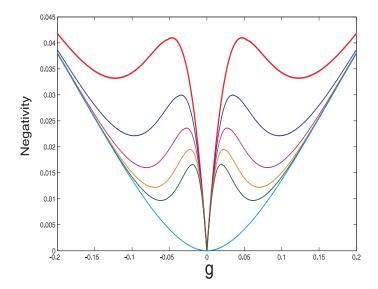


Figure 5: (Color Online) The entanglement of adjacent spins as measured by their negativity for different values of system size, from top to bottom equal to $N = 15, 20, 25, 30, 35, \infty (\sigma = 1)$.

linear superposition of all states. We consider the cases $\sigma = 1$ and $\sigma = -1$ separately.

Case a: $\sigma=1$ Near the critical point $|g|\ll 1$, the dominant amplitudes are $\psi_{00\cdots 0}$, $\psi_{k,\overline{l}}$ and $\psi_{\overline{k},l}$, where $\psi_{k,\overline{l}}$ denotes the amplitude of a state $|k,\overline{l}\rangle$ in which two spins at sites k and l are respectively excited to 1 and -1. To first order in g, the state (1) becomes

$$|\Psi\rangle \simeq |0, 0, \cdots 0\rangle - g \sum_{k < l=1}^{N} (|k, \overline{l}\rangle + |\overline{k}, l\rangle).$$
 (25)

It is not difficult to find the negativity of this state which is

$$\mathcal{E}(\rho(1,r)) = 2|g|. \tag{26}$$

Case b: σ =-1, N=even For $|g| \ll 1$, the dominant amplitudes are $\psi_{00\cdots 0} \propto 1$ and

$$\psi_{k\overline{l}} \propto tr(A_0^{k-1}A_1A_0^{l-k-1}A_{\overline{1}}A_0^{N-l}) \propto tr(A_1A_0^{l-k-1}A_{\overline{1}}) = 2g(-1)^{l-k},$$

and $\psi_{\overline{k},l} \propto 2g(-1)^{k-l}$, where we have used the fact that $A_0^m A_1 = A_1$ for any m. Thus the state becomes

$$|\Psi\rangle \simeq |0, 0, \cdots 0\rangle + g \sum_{k < l} (-1)^{l-k} (|k, \overline{l}\rangle + |\overline{k}, l\rangle).$$
 (27)

From the two-site density matrix, one can find the negativity which is given by $\mathcal{E}(\rho(1,r)) = 2|g|$.

Case c: σ =-1, N=odd In this case we have $\psi_{00\cdots 0} = 0$ and so the state becomes

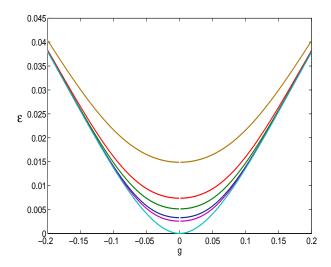


Figure 6: (Color Online) The entanglement of adjacent spins as measured by their negativity for different systems with odd-size, from top to bottom equal to $N = 15, 21, 25, 31, 35, (\sigma = -1)$.

$$|\Psi\rangle = \frac{1}{\sqrt{N(N-1)}} \sum_{k < l} (-1)^{l-k} \left(|k, \overline{l}\rangle - |\overline{k}, l\rangle \right). \tag{28}$$

The entanglement of such a state can be calculated by determining the reduced density matrix of two sites. After a rather lengthy but calculation, we find

$$\mathcal{E}(\rho(1,r)) = \frac{1}{2N(N-1)}|(N-2)(N-3)-1$$
$$-\sqrt{[(N-2)(N-3)+1]^2 + 8(N-2)^2}|.$$

6 Conclusion

In this paper we have studied in detail, the entanglement properties of a general one-parameter family of matrix product state of spin one particles defined on a ring. The state has some plausible symmetries, like rotational symmetry around the z, axis, and the parity symmetry. Such a state is the ground state of local Hamiltonian defining the nearest neighbor interaction of spins. The state goes a sharp transition when its continuous parameter, denoted by g, passes through a critical point. In the thermodynamic limit, this can be ascribed to a quantum phase transition of the system described by the local Hamiltonian. We have studied the entanglement properties of the state, near this transition point, both in the thermodynamic limit and for finite chains. The study of finite chains is motivated by the possible role of such systems in quantum computing. We have considered two measures of entanglement, namely the entanglement of one site with the rest of the chain, (also studied in [18] for infinite

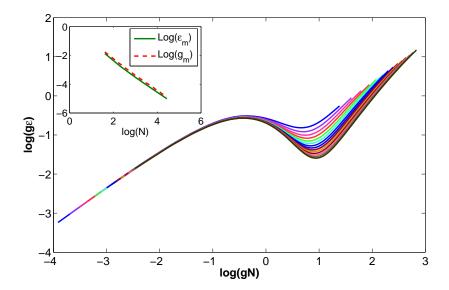


Figure 7: (Color Online) Near the critical point, the entanglement of nearest neighbor sites, as measured by their negativity, shows a scaling behavior. The main plot shows $\log(N\mathcal{E})$ versus $\log(gN)$, for values of $N=20,25,30\cdots 80,85$. All the data collapse on one single curve, for a range of 4 orders of magnitude in the value of gN.

rings) and the entanglement of two distant sites with each other. Our findings can be summarized as follows: 1- When the parameter g approaches its critical value $g_0 = 0$, the range of entanglement (between distant spins) increases indefinitely at the cost of its magnitude, fig. (1), 2- for any non-zero value of g, there is a threshold distance beyond which there is no entanglement between spins, fig. (3), 3- for any two spins with a fixed distance, there is a threshold system size, above which entanglement vanishes, fig. (4), and finally there is a kind of scaling behavior in entanglement properties of neighboring sites, near the critical point (Figure 7).

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